A NEW SECOND-ORDER ITERATION METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract. The fixed point iterative method has first order convergence. The method described here by the name My Method One(MMO), has the convergence order two which is same as of New Iteration Method(NIM). It is faster than the fixed point method and in some examples it is faster than NIM. Comparison table demonstrates the faster convergence of My Method One(MMO)
.

1. Introduction

Solving equations in one variable is the most discussed problem in numerical analysis. Their are several numerical techniques for solving nonlinear equations (see for example [[2]- [4]] and the references there in). Fixed point iteration method [7] is the fundamental algorithm for solving nonlinear equations in one variable. It has first order convergence. The methods given by [9], [11] and [12] are efficient methods for solving non-linear equations. T he method given by [10] is a fast method and take two evaluations. It is always need to find more efficient method for olving non-linear equations.

A function f is given and we have to find at least one solution to the equation $f\ (x) = 0$. Note that, priorly, we do not put any restrictions on the function f, we need to be able to evaluate the function, otherwise,we can not even check that a given solution is true, that is, $f(\alpha) = 0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of "good behavior." The more we assume, the more potential we have, on the one hand, to develop fast algorithms for finding the root. At the same time, the more we assume, the fewer the functions are going to satisfy our assumptions! This is a fundamental paradigm in numerical analysis. We know that one of the fundamental algorithm of solving nonlinear equations is so called fixed point iteration method. In this method equation is rewritten as

$$x = g(x), \tag{1.1}$$

where

(i) there exist $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$,

(ii) there exist $[a, b]$ such that $|g'(x)| \le L < 1$ for all $x \in [a, b]$.

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Theorem 1.1. *Let $x_{k+1} = \phi_{r+1}(x_k)$ $(k = 0, 1, 2, \cdots)$ be an iterative method of order r fpr finding a simple or multiple root of a given function f (sufficiently many times differentiable). Then the iterative method defined by*

$$x_{k+1} = \phi_{r+1}(x_k) = x_k - \frac{x_k - \phi_r(x_k)}{1 - \frac{1}{r}\phi'_r(x_k)}, r \geq 2, k = 0, 1, 2, \cdot\cdot$$

Theorem 1.2. [3] *Suppose that $g \in C^p[a, b]$. If $g^{(k)}(x) = 0$ for $k = 1, 2, ..., p-1$ and $g^{(p)}(x) \neq 0$, then the sequence $\{x_n\}$ is of order p.*

Algorithm 1. {[9] *New Iterative Method* }*For a given x_0 , Shin et al gives the approximate solution x_{n+1} by an iteration scheme*

$$x_{n+1} = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)}$$

This scheme has converges order 2.

2. Main Result

In the fixed-point iteration method for solving the nonlinear equation $f(x) = 0$, the equation can be rewritten as

$$x = g(x) \quad (2.1)$$

Let α be the root of $f(x) = 0$ or of $x = g(x)$, we can write $x = g(x)$ as

$$0 = g(x) - x$$

adding $x(x - g(x)) + xh(x)$ to both sides ,

$$x(x - g(x)) + xh(x) = x(x - g(x)) + xh(x) + g(x) - x$$

from here we will get

$$x = \frac{x(x - g(x)) + xh(x) + g(x) - x}{x - g(x) + h(x)} = H_h(x) \quad (2.2)$$

In order to 2.2 to be efficient we choose h(x) such that $H'_h(\alpha) = 0$
i.e

$$\frac{-1 + h(\alpha) + g'(\alpha)}{h(\alpha)} = 0$$

gives

$$h(\alpha) = 1 - g'(\alpha)$$

From we suggest to take $h(x) = 1 - g'(x)$, put this value in 2.2, we have

$$x = \frac{x(x - g(x)) + x(1 - g'(x)) + g(x) - x}{x - g(x) + 1 - g'(x))}$$

on simplification it gives

$$x = \frac{x(x - g(x)) - xg'(x) + g(x)}{x - g(x) + 1 - g'(x))}$$

This formulation allows us to suggest the following iteration methods for solving nonlinear equation 2.1

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Algorithm 2.

For a given x_0 we calculate the approximation solution x_{n+1}, by the iteration scheme

$$x_{n+1} = \frac{x_n(x_n - g(x_n)) + g(x_n) - x_n g'(x_n)}{x_n - g(x_n) + 1 - g'(x_n)}$$

3. Convergence Analysis of My Method One

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D and consider the nonlinear equation $f(x) = 0$(or $x = g(x)$) has a simple root $\alpha \in D$, where $g : (x) : D \subset \mathbb{R} \to \mathbb{R}$ be sufficiently smooth in the neighborhood of the root α ; then the order of convergence of algorithm is at least 2.

Proof. To analysis the convergence of My Method One, consider the functional equation

$$x_{n+1} = \frac{x_n(x_n - g(x_n)) + g(x_n) - x_n g'(x_n)}{x_n - g(x_n) + 1 - g'(x_n)} \;,\; x_n - g(x_n) + 1 - g'(x_n) \neq 0 \tag{3.1}$$

We can write 3.1 as

$$x_{n+1} = x_n - \frac{x_n - g(x_n)}{x_n - g(x_n) + 1 - g'(x_n)} \tag{3.2}$$

Let α be the root of $f(x) = 0$(or $g(\alpha) = \alpha$), and $x_n = \epsilon_n + \alpha$, $x_{n+1} = \epsilon_{n+1} + \alpha$. Then applying Taylor's series

$$\begin{aligned} g(x_n) &= g(\alpha) + \epsilon_n g'(\alpha) + \frac{\epsilon_n^2}{2} g''(\alpha) + \bigcirc(\epsilon_n^3) \\ g'(x_n) &= g'(\alpha) + \epsilon_n g''(\alpha) + \bigcirc(\epsilon_n^2) \end{aligned}$$

Let $k_n = \frac{g^{(n)}(\alpha)}{n!(1-g'(\alpha))}$, then

$$\begin{aligned} x_n - g(x_n) &= (1 - g'(\alpha))(\epsilon_n - k_2\epsilon_n^2 + \bigcirc(\epsilon_n^3)), \text{ and} \\ 1 - g'(x_n) &= 1 - g'(\alpha) - 2k_2\epsilon_n + \bigcirc(\epsilon_n^2) \\ x_n - g(x_n) + 1 - g'(x_n) &= (1 - g'(\alpha))(1 + (1 - 2k_2)\epsilon_n + \bigcirc(\epsilon_n^2)) \end{aligned}$$

Subtitute these values in 3.2, then we have

$$\begin{aligned} \epsilon_{n+1} &= \epsilon_n - (\epsilon_n - k_2\epsilon_n^2 + \bigcirc(\epsilon_n^3))(1 + (1 - 2k_2)\epsilon_n + \bigcirc(\epsilon_n^2))^{-1}, \\ \epsilon_{n+1} &= \epsilon_n - (\epsilon_n - k_2\epsilon_n^2 + \bigcirc(\epsilon_n^3))(1 - (1 - 2k_2)\epsilon_n + \bigcirc(\epsilon_n^2)), \\ \epsilon_{n+1} &= (1 - k_2)\epsilon_n^2 + \bigcirc(\epsilon_n^3) \end{aligned}$$

Hence Algorithm 2 has convergence order 2. □

Comparison 1. *Comparison of Fixed Point Method(FPM), New Iteration Method(NIM) and My Method One(MMO) is shown in the Table correct up to eight decimal places*

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Table: Comparison of FPM, NIM, MM1				
Method	N	N_f	$f(x_n)$	x_n
$f(x) = x3 + x2 - 2 = 0$, $g(x) = 2/x^2 - 1$				
$x_0 = 6$				
FPM	174	174	2.142359e-10	1.00000000
NIM	169	338	2.281965e-18	1.00000000
MM1	10	20	3.420668e-09	1.00000000
$f(x) = x - \cos x$, $g(x) = \cos x$				
$x_0 = 10$				
FPM	49	49	9.952575e-10	0.73908513
NIM	diverges	not applied	not applied	not applied
MM1	16	32	1.269393e-07	0.73908513
$f(x) = x^3 + 4x^2 + 8x + 8$, $g(x) = -1 - 1/2x^2 - 1/8x3$				
$x_0 = -1.7$				
FPM	28	28	5.940237e-09	-2.00000000
NIM	4	8	2.503802e-13	-2.00000000
KMO	4	8	2.180891e-06	-2.00000000
$f(x) = xe^{x^2} - \sin x + 3\cos x - x + 5$, $g(x) = xe^{x^2} - \sin x + 3\cos x + 5$				
$x_0 = 0.5$				
FPM	diverges	not applied	not applied	not applied
NIM	64	128	3.850191e-13	-1.26464680
MM1	4	8	3.375573e-09	-1.26464680
$f(x) = x^2 \tan x - 1$, $g(x) = \sqrt{\frac{1}{\tan x}}$				
$x_0 = 0.01$				
FPM	247	247	2.196353e-09	0.89520605
NIM	8	16	9.407818e-18	0.89520605
MM1	7	14	4.864718e-06	0.89520605
$f(x) = x^2 - 5x + 6$, $g(x) = \frac{(x^2+6)}{5}$				
$x_0 = 1.5$				
FPM	81	81	4.241609e-09	2.00000000
MIM	5	10	5.396595e-16	2.00000000
MM1	3	6	1.124858e-03	2.00000000

Conclusion 1. *A new second order iterative method (MMO) has been established. From table, we can conclude that*

1. *The MMO has convergence order 2, which is equal to New iteration method.*
2. *The MMO is performing very well in comparison to FPM.*

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